A CHARACTERIZATION OF NORMAL SUBGROUPS VIA N-CLOSED SETS

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ABSTRACT. Let $(G,*)$ be a semigroup, $D \subseteq G$, and $n \geq 2$ be an integer. We say that $(D,*)$ *is an n-closed subset of* G if $a_1 * \cdots * a_n \in D$ for every $a_1, ..., a_n \in D$. Hence every closed set is a 2-closed set. The concept of n-closed sets arise in so many natural examples. For example, let D be the set of all odd integers, then $(D,+)$ is a 3-closed subset of $(\mathbb{Z},+)$ that is not a 2-closed subset of $(\mathbb{Z},+)$. If $K = \{1,4,7,10,...\}$, then $(K,+)$ is a 4-closed subset of $(\mathbb{Z},+)$ that is not an n-closed subset of $(\mathbb{Z},+)$ for $n = 2,3$. In this paper, we show that if $(H,*)$ is a subgroup of a group $(G,*)$ such that $[H:G] = n < \infty$, then H is a normal subgroup of G if and only if every left coset of H is an $n+1$-closed subset of G.

1. INTRUDUCTION

In this paper, we introduce the concept of n-closed sets for some integer $n \geq 2$. Let $(G,*)$ be a semigroup, $D \subseteq G$, and $n \geq 2$ be an integer. We say that $(D,*)$ *is an n-closed subset of* G if $a_1 * \cdots * a_n \in D$ for every $a_1, ..., a_n \in D$. Hence every closed set is a 2-closed set. The concept of n-closed sets arise in so many natural examples. For example, let D be the set of all odd integers, then $(D,+)$ is a 3-closed subset of $(\mathbb{Z},+)$ that is not a 2-closed subset of $(\mathbb{Z},+)$. If $K = \{1,4,7,10,...\}$, then $(K,+)$ is a 4-closed subset of $(\mathbb{Z},+)$ that is not an n-closed subset of $(\mathbb{Z},+)$ for $n = 2,3$. In the second section of this paper paper, many basic properties of n-closed sets are studied. For example, we show that if a finite set D of a group $(G,*)$ is an n-closed subset of G, then D is a left coset of a subgroup of G. In the third section, we give a characterization of normal subgroups via n-closed sets. For example, we show that if $(H,*)$ is a subgroup of a group $(G,*)$ such that $[H:G] = n < \infty$, then H is a normal subgroup of G if and only if every left coset of H is an $n+1$-closed subset of G. Though we feel that the proofs of many results in this short paper are elementary, we feel that the whole idea is original and it has not been considered in the literature.

Let $(G,*)$ be a group. If H is a subset of G and $H \neq G$, then we write $H \subset G$. If H is a subgroup of G, then $[H:G]$ denotes the number of all distinct left cosets of H. If $a \in G$ and $n \geq 1$ is an integer, then $a^n = a * \cdots * a$ (n times), $(a^n)^{-1}$ is the inverse of a^n in G, and $\mid a \mid$ denotes the order of a in G. Let D be a subset of G, and $d_1, ..., d_k \in D$. Then $d_1 * \cdots * d_k * D = \{d_1 * \cdots * d_k * d \mid d \in D\}$. As usual, $\mathbb{R}, \mathbb{Q}$, and $\mathbb{Z}$ will denote real numbers, rational numbers, and integers, respectively.

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2. Basic properties of n-closed sets

Theorem 2.1. *Let D be a finite subset of a group $(G, *)$. Suppose that D is not a 2-closed subset (subgroup) of G, and for some integer $n \geq 3$, $(D, *)$ is an n-closed subset of $(G, *)$. Then*

(1). *$(d_1 * \cdots * d_{n-2} * D, *)$ is a subgroup of G for every $d_1, ..., d_{n-2} \in D$. In particular, $(d^{n-2} * D, *)$ is a subgroup of G for every $d \in D$.*

(2). *Let $b, d_1, ..., d_{n-2} \in D$. Then $D = b * d_1 * \cdots * d_{n-2} * D$ (i.e., D is a left coset of the subgroup $(d_1 * \cdots * d_{n-2} * D, *)$ of $(G, *)$). In particular, $b * d^{n-2} * D = D$ for every $b, d \in D$.*

(3). *$d^{n-2} * D = b^{n-2} * D = d_1 * \cdots * d_{n-2} * D$ for every $d, b, b_1, ..., b_{n-2} \in D$.*

Proof. Suppose that D is an n-closed subset of G for some integer $n \geq 3$.

(1). Let $d_1, ..., d_{n-2} \in D$. Since D is a finite subset of the group $(G, *)$, we only need to show that $(d_1 * \cdots * d_{n-2} * D, *)$ is a 2-closed subset of G. Let $a, b \in d_1 * \cdots * d_{n-2} * D$. Hence $a = d_1 * \cdots * d_{n-2} * h_1$ and $b = d_1 * \cdots * d_{n-2} * h_2$ for some $h_1, h_2 \in D$. Since D is an n-closed subset of G, $h_1 * d_1 * \cdots * d_{n-2} * h_2 = c \in D$, and thus $a * b = d_1 * \cdots * d_{n-2} * h_1 * d_1 * \cdots * d_{n-2} * h_2 = d_1 * \cdots * d_{n-2} * c \in d_1 * \cdots * d_{n-2} * D$.

(2). Let $b, d_1, ..., d_{n-2} \in D$. Since D is an n-closed subset of G, $b * d_1 * \cdots * d_{n-2} * a \in D$ for every $a \in D$. Since D is a finite subset of G and $b * d_1 * \cdots * d_{n-2} * a_1 = b * d_1 * \cdots * d_{n-2} * a_2$ for some $a_1, a_2 \in D$ if and only if $a_1 = a_2$, we conclude that $D = b * d_1 * \cdots * d_{n-2} * D$.

(3). $d, b, b_1, ..., b_{n-2} \in D$. Since $D = b * d^{n-2} * D = b * b^{n-2} * D = b * d_1 * \cdots * d_{n-2} * D$ by (2), we conclude that $d^{n-2} * D = b^{n-2} * D = d_1 * \cdots * d_{n-2} * D$ for every $d, b, b_1, ..., b_{n-2} \in D$. □

Corollary 2.2. *Let $(G, *)$ be a finite group and D be a subset of G. Suppose that D is not a 2-closed subset (subgroup) of G, and for some integer $n \geq 3$, $(D, *)$ is an n-closed subset of $(G, *)$. Then $H = d^{n-2} * D$ is a subgroup of G for every $d \in D$ and D is a left coset of H.*

The following example shows that the hypothesis that D is finite in Theorem 2.1 is crucial.

Example 2.3. *Let $G = (\mathbb{Z}, +)$, and $D = \{1, 3, 5, ...,\}$ be the set of all positive odd numbers of $\mathbb{Z}$. Then D is a 3-closed subset of $\mathbb{Z}$, but $(a^{3-2} + D = a + D, +)$ is not a subgroup of G for every $a \in D$.*

In view of the proof of Theorem 2.1, we have the following.

Corollary 2.4. *Let $(G, *)$ be a a semigroup and $(D, *)$ be a an n-closed subset of $(G, *)$ for some integer $n \geq 3$. Then $(d_1 * \cdots * d_{n-2} * D), *)$ is a 2-closed subset of $(G, *)$ for every $d_1, ..., d_{n-2} \in D$. In particular, $(d^{n-2} * D, *)$ is a 2-closed subset of $(G, *)$ for every $d \in D$.*

Theorem 2.5. *Let $(G, *)$ be a group, $H \subset G$ be a subgroup of G, and $L = a * H$ for some $a \in G \setminus H$. Suppose that $(L, *)$ is an n-closed subset of G for some integer $n \geq 2$, and let $k \geq 2$ be the least integer such that $(L, *)$ is a k-closed subset of G. Then:*

(1). *$a^{n-1} \in H$, and hence $n \geq 3$.*

(2). *$a * H = H * a = L$, and hence $b * H = H * b = L$ for every $b \in L$.*

(3). *$a^{n-2} * L = L * a^{n-2} = H$, and hence $b_1 * \cdots * b_{n-2} * L = L * b_1 * \cdots * b_{n-2} = H$ for every $b_1, ..., b_{n-2} \in L$.*

(4) *$a^m \in H$ for some integer $m > 0$ if and only if $(k-1) \mid m$, and hence for every $d \in L$, $d^m \in H$ for some positive integer m if and only if $(k-1) \mid m$.*

(5). *$(L, *)$ is an m-closed subset of G for some positive integer m if and only if $m = c(k-1)+1$ for some integer $c \geq 1$.*

Proof. **(1)**. Since L is an n-closed subset of G and $a \in L$, $a^n = a * h \in L$ for some $h \in H$, and thus $a^{n-1} = h \in H$. Since $a^{n-1} \in H$ and $a \in G \setminus H$, we have $n \geq 3$.

(2). Let $h \in H$. We show that $a*h = h_1*a$ for some $h_1 \in H$. Since $a^{n-1} \in H$ by (1), $h_2 = (a^{n-1})^{-1} \in H$. Since L is n-closed, $(a*h)*(a*h*h_2)*a^{n-2} = a*h_3 \in L$ for some $h_3 \in H$. Thus $(a*h*h_2)*a^{n-2} = h^{-1}*h_3$, and hence $(a*h*h_2)*a^{n-1} = h^{-1}*h_3*a$. Since $h_2 = (a^{n-1})^{-1}$, we have $h^{-1}*h_3*a = (a*h*h_2)*a^{n-1} = a*h$. Since $h_1 = h^{-1}*h_3 \in H$, $h_1*a = a*h$. Thus $a*H = H*a$. Let $b \in L$. We show that $b*H = H*b = L$ for every $b \in L$. Since $b \in L$, $b = a*h$ for some $h \in H$. Since $a*H = H*a$, $a*h = h_1*a$ for some $h_1 \in H$. Thus $L = (a*h)*H = a*H = H*a = H*(h_1*a) = H*(a*h)$.

(3). Since $a^{n-1} \in H$ by (1), we have $a^{n-2}*L = a^{n-1}*H = H$. Since $a*H = H*a$ by (2), we have $H = a^{n-2}*L = a^{n-1}*H = H*a^{n-1} = (H*a)*a^{n-2} = L*a^{n-2}$. Let $b_1, ..., b_{n-2} \in L$. Since $a*H = H*a$, we have $b_1*\cdots*b_{n-2} = a^{n-2}*h$ for some $h \in H$. Thus $b_1*\cdots*b_{n-2}*L = a^{n-1}*H = H*a^{n-1} = H*b_1\cdots*b_{n-2} = H$.

(4). Suppose that $(k-1) \mid m$ for some positive integer m. Since $a^{k-1} \in H$ by (1), we have $a^m \in H$. Conversely, suppose that $a^m \in H$ for some integer $m > 0$. Then $m = b(k-1)+r$ for some integers $b, r \geq 0$ such that $0 \leq r < (k-1)$. We show that $r = 0$. Hence $a^m = a^{b(k-1)+r} = a^{b(k-1)}*a^r \in H$. Since $a^{k-1} \in H$ and $a^{b(k-1)}*a^r \in H$, we have $a^r \in H$. Let $d_1, ..., d_{r+1} \in L = a*H$. Since $a*H = H*a$ by (2) and $a^r \in H$, there is an $h \in H$ such that $d_1*\cdots*d_{r+1} = a^{r+1}*h = a*a^r*h \in a*H = L$. Thus L is an $r+1$-closed subset of G, which is a contradiction since $r+1 \leq (k-1)$ and $m \neq 0$. Hence $r = 0$ and $b \geq 1$.

(5). Suppose that L is an m-closed subset of G for some positive integer m. Then $a^{m-1} \in H$ by (1). Hence $m-1 = c(k-1)$ for some integer $c \geq 1$ by (4), and thus $m = c(k-1)+1$. Conversely, suppose that $m = c(k-1)+1$ for some integer $c \geq 1$. Let $d_1, ..., d_m \in L$. Since $a*H = H*a$ by (2) and $a^{m-1} \in H$ by (1), there is an $h \in H$ such that $d_1*\cdots*d_m = a^m*h = a*a^{m-1}*h \in a*H = L$. Thus L is an m-closed subset of G.

□

In light of Theorem 2.5[(1) and (2)] and the proof of Theorem 2.5(5), we have the following corollary.

Corollary 2.6. *Let $(G, *)$ be a group, $H \subset G$ be a subgroup of G, and $L = a*H$ for some $a \in G \setminus H$. Let $n \geq 3$. Then $(L, *)$ is an n-closed subset of G if and only if $a*H = H*a$ and $a^{n-1} \in H$.*

The proof of the following lemma is similar to the proof of the well-known fact: Let $(G, *)$ be a group and $a \in G$ such that $\mid a \mid = n < \infty$, then $\mid a^m \mid = n/gcd(m,n)$ for every integer $m > 0$. Hence we omit the proof.

Lemma 2.7. *Let $(G, *)$ be a group, $H \subset G$ be a subgroup of G, and $a \in G \setminus H$. Suppose that $a^n \in H$ for some integer $n \geq 2$, and let $k \geq 2$ be the least integer such that $a^k \in H$. Then for each $m \geq 1$, we have $c = k/gcd(m,k)$ is the least positive integer such that $(a^m)^c \in H$. Furthermore, $(a^m)^f \in H$ for some integer $f \geq 1$ if and only if $c \mid f$.*

Theorem 2.8. *Let $(G, *)$ be a group, $H \subset G$ be a subgroup of G, and $L = a * H$ for some $a \in G \setminus H$. Suppose that $(L, *)$ is an n-closed subset of G for some integer $n \geq 3$, and let $k \geq 3$ be the least integer such that $(L, *)$ is a k-closed subset of G. For each integer $m \geq 1$, let $c = (k-1)/gcd(m, k-1)$. Then $a^m * H$ is a $c+1$-closed subset of G. Furthermore, $a^m * H$ is an f-closed subset of G for some integer $f \geq 1$ if and only if $f = bc + 1$ for some integer $b \geq 1$.*

Proof. Let $m \geq 1$, $K = a^m * H$, and $c = (k-1)/gcd(m, k-1)$. Since $L = a * H$ is an n-closed subset of G for some $n \geq 3$, we have $a * H = H * a$ by Theorem 2.5 and thus $K = a^m * H = H * a^m$. Since $k-1$ is the smallest integer such that $a^{k-1} \in H$, we have $c = (k-1)/gcd(m, k-1)$ is the smallest integer such that $(a^m)^c \in H$ by Lemma 2.7. Hence $K = a^m * H$ is a $c+1$-closed subset of G by Corollary 2.6. Thus $K = a^m * H$ is an f-closed subset of G for some integer $f \geq 2$ if and only $f = bc+1$ by Theorem 2.5(5) □

The following is a trivial example of n-closed sets.

Example 2.9. *Let $(G, *)$ be a group with at least two elements, and let a be a non-identity element of G. Suppose that $\mid a \mid = k < \infty$ for some integer $k \geq 2$. Then $\{a\}$ ia an m-closed subset of G for some $m \geq 3$ if and only if $m = bk + 1$ for some integer $b \geq 1$.*

It is possible to have a group $(G, *)$ and a left coset L of a subgroup H of G such that for some integer $n \geq 2$, $a^n \in H$ and $a^{n+1} \in L$ for every $a \in L$, but yet L is not an m-closed subset of G for every integer $m \geq 2$. We have the following example.

Example 2.10. *Let $G = S_3$ be the permutation group on 3 elements. Then $H = \{(1), (1\ \ 2)\}$ is a subgroup of G, and $L = (1\ \ 3)oH = \{(1\ \ 3), (1\ \ 2\ \ 3)\}$ is a left coset of H. Then $a^6 \in H$ and $a^7 \in L$ for every $a \in L$. Since $L = (1\ \ 3)oH = \{(1\ \ 3), (1\ \ 2\ \ 3)\} \neq Ho(1\ \ 3)$, L is not an m-closed subset of G for every integer $m \geq 2$ by Corollary 2.6.*

It is possible to have a group $(G, *)$ and a subgroup H of G such that for each $n \geq 3$, there is a left coset of H that is an n-closed subset of G, but it is not an m-closed subset for each integer m, $2 \leq m < n$. We have the following example.

Example 2.11. *Let $G = (\mathbb{Q}, +)$. Then $H = (\mathbb{Z}, +)$ is a subgroup of G. Let $n \geq 3$. Then $L = \frac{1}{n-1} + \mathbb{Z}$ is a left cost of H that is an n-closed subset of G, but it is not an m-closed subset for each integer m, $2 \leq m < n$.*

It is possible to have a subgroup H of a group G and a left coset $L = a * H$ for some $a \in G \setminus H$ such that $a * H = H * a$, but L is not an n-closed subset of G for every $n \geq 2$. We have the following example.

Example 2.12. *Let $G = (\mathbb{R}, +)$. Then $H = (\mathbb{Z}, +)$ is a subgroup of G, $L = \sqrt{2} + \mathbb{Z}$ is a left coset of H, and $\sqrt{2} + \mathbb{Z} = \mathbb{Z} + \sqrt{2}$, but $(L, +)$ is not an n-closed subset of G for each integer $n \geq 2$.*

3. A characterization of normal subgroups

In view of Corollary 2.6, we have the following characterization of normal subgroups via n-closed subsets.

Theorem 3.1. *Let $(G, *)$ be a group and $H \subset G$ be a subgroup of G. The following statements are equivalent:*

(1). *H is a normal subgroup of G and for each $a \in G \setminus H$, there is an integer $n \geq 2$ such that $a^n \in H$.*

(2). *For each $a \in G \setminus H$, there is an integer $m \geq 3$ such that $a * H$ is an m-closed subset of G.*

Theorem 3.2. *Let $(G, *)$ be a group and $H \subset G$ be a subgroup of G. Suppose that $[H : G] = n < \infty$. Then H is a normal subgroup of G if and only if $a * H$ is an $n + 1$-closed subset of G for each $a \in G \setminus H$.*

Proof. Suppose that H is a normal subgroup of G. Since G/H is a group with exactly n distinct elements, we have $a^n \in H$ for each $a \in G \setminus H$. Thus we are done by Corollary 2.6. $\square$

Corollary 3.3. *Let $(G, *)$ be a finite group, $H \subset G$ be a subgroup of G, and $n = [H : G]$. Then H is a normal subgroup of G if and only if $a * H$ is an $n + 1$-closed subset of G for each $a \in G \setminus H$.*

References

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